Calculating a Guitar’s Surface Area with Bézier Curves

Mete Yazar

Introduction

The Pen tool and numerous other spline functions in contemporary applications utilize the Bezier model. A mathematical spline is defined as a curve that passes through a given set of points and maintains a certain number of continuous derivatives. Historically, splines were first used in ship construction to create outline curves for the main structures, crafted by hand without the use of technical mathematical formulas. However, during the early 1960s, the French car manufacturer Citroen employed a young mathematician, Paul de Casteljau, to address various geometrical and practical issues arising from car production lines. De Casteljau developed a system primarily focused on the design of curves and surfaces, rather than merely reproducing existing blueprints. Concurrently, engineer Pierre Bezier, working in the design department of Renault, a rival French car manufacturer, recognized the necessity for mathematical representations of mechanical parts.

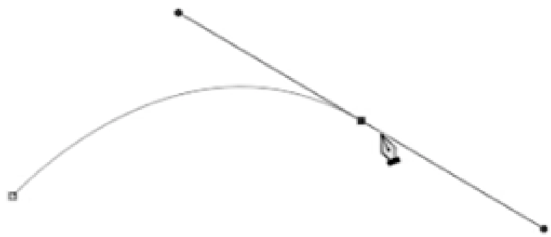

Figure 1: Pen tool in Photoshop.

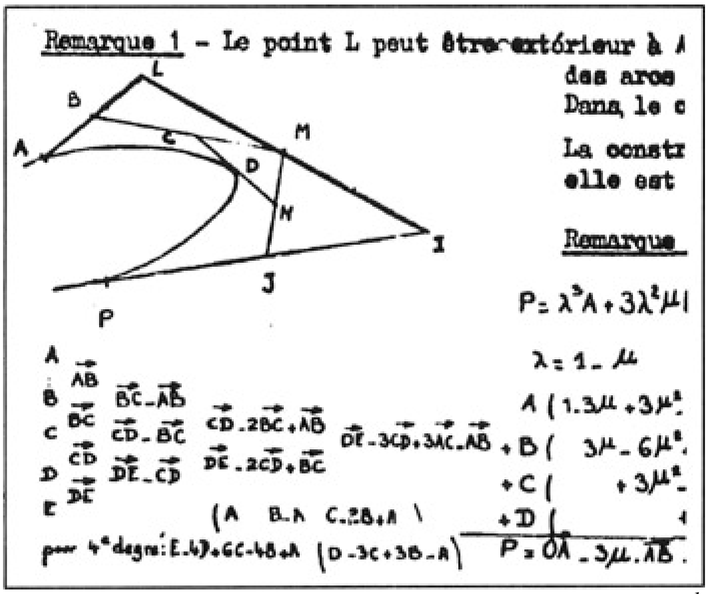

Figure 2: An excerpt from P. de Casteljau's writings.[1]

Their works remained independent from each other, but the breakthrough insight of this realization was the use of control polygons, a technique that was never used before. Instead of defining a curve through points on it, a control polygon utilizes points near it. This meant, that

[1] Farin, Gerald E.. “A History of Curves and Surfaces in CAGD.” Handbook of Computer Aided Geometric Design (2002).

instead of changing the curve directly, one changes the control polygon, and the curve follows in a very intuitive way. Bezier curves can use functions depending on how many points are going to be used to create control polygons. Thus, amplifying the needed function to create a spline.

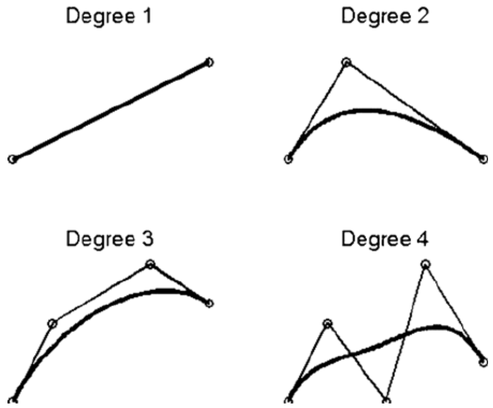

Figure 3: Various degrees of Bezier curves.

Formulating Degrees

A Bezier curve is constructed by points. These points are called control points. Every Bezier curve is a series of linear interpolations between control points. Let P_0 and P_1 be two control points in a two-dimensional cartesian space. The line between P_0 and P_1 can be defined by linear interpolations. First, a vector V_0 is defined with its tail on the origin and tip on P_0. Then a second vector V_1 is defined with the same origin as P_1. The curve that is going to be defined is a parametric curve. Therefore, a parameter t is going to be used as a real number between 0 and 1. V_0 is going to be multiplied by t. This is a vector scalar multiplication which means the length of V_0 decreases without a change in direction. V_1 is going to be multiplied by 1-t. Then, the resulting vectors are added. This addition will generate a vector in which its tip point Q is on the line between the line P_0 and P_1. If t is 0 then x is on the P_0 or if t is 1 then x is on P1. All real numbers between 0 and 1 will generate a point on the line. The figure below illustrates this. This is called interpolation. Let Q(x,y) be a point on the line between P_0 and P_1. The functions to calculate x and y are;

$$f(x) = tP_{0x} + (1-t)P_{1x}$$

$$g(y) = tP_{0y} + (1-t)P_{1y}$$

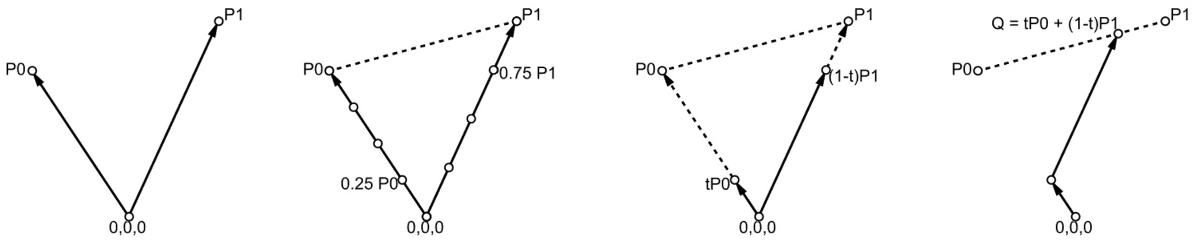

Figure 4: First-degree (linear) functions

The Functions above are parametric functions of a linear (degree 1) Bezier curve. Second-degree Bezier curves have three control points P_0, P_1, and P_2. To draw a second-degree (quadratic) Bezier curve the linear interpolation procedure explained above is repeated 3 times. The first parameter t is decided between 0 and 1. Q_0 is defined as a linear interpolation between

P_0 and P_1. Then Q_1 will be defined as another linear interpolation between P_1 and P_2. Point X on the curve is a linear interpolation between Q_0 and Q_1.

$$Q_0 = tP_0 + (1-t)P_1$$

$$Q_1 = tP_1 + (1-t)P_2$$

$$X = tQ_0 + (1-t)Q_1$$

Therefore, the equation can be combined and put in a single formula;

$$X = t(tP_0 + (1-t)P_1) + (1-t)(tP_1 + (1-t)P_2)$$

$$X = t^2P_0 + 2(1-t)tP_1 + (1-t)^2P_2$$

This formula is used to find X with control points without the need to calculate Q_0 and Q_1.

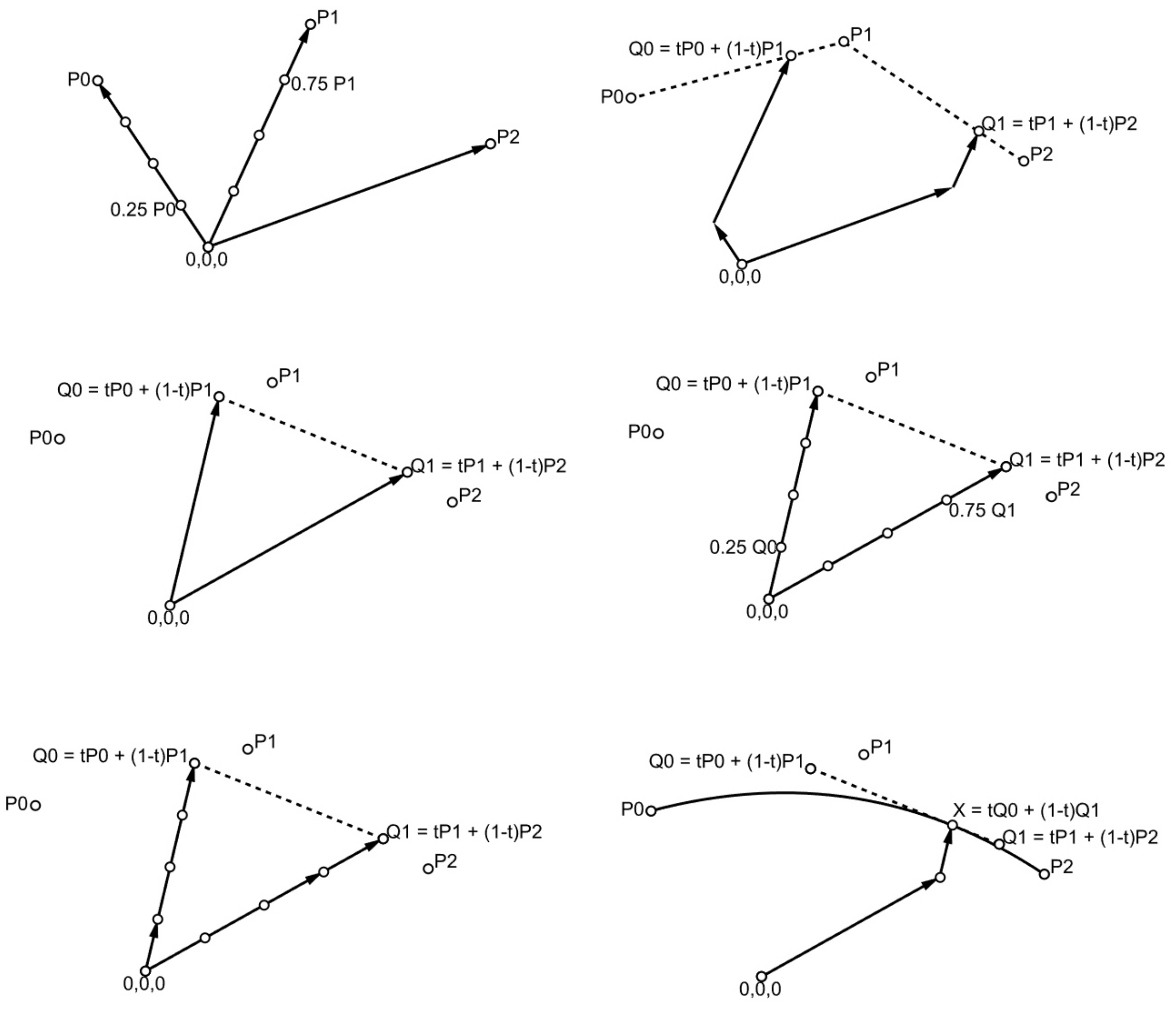

Figure 5: Degree 2 Bezier curve construction.

Third Degree (Cubic) is the most common type of Bezier curve used in CAD because it generates smooth curves with a minimum number of control points. A third-degree Bezier curve has four control points P_0, P_1, P_2, and P_3. To draw a Third-degree Bezier curve the linear interpolation procedure is repeated 6 times. The first parameter t is decided between 0 and 1. Q_0 is defined as a linear interpolation between P_0 and P_1. Then Q_1 will be defined as another linear interpolation between P_1 and P_2. Then Q_2 will be defined as another linear interpolation between P_2 and P_3. R_0 and R_1 will be defined as another linear interpolation between Q_0, Q_1 and Q_1, Q_2 respectively. Point X on the curve is a linear interpolation between R_0 and R_1.

$$Q_0 = tP_0 + (1-t)P_1$$

$$Q_1 = tP_1 + (1-t)P_2$$

$$Q_2 = tP_2 + (1-t)P_3$$

$$R_0 = tQ_0 + (1-t)Q_1$$

$$R_1 = tQ_1 + (1-t)Q_2$$

$$X = tR_0 + (1-t)R_1$$

The above equation is going to be used while defining the guitars' shape. This equation also can be combined and put in a single formula.

$$X = (1-t)^3P_0 + 3(t-1)^2tP_1 + 3(1-t)t^2P_2 + t^3P_3$$

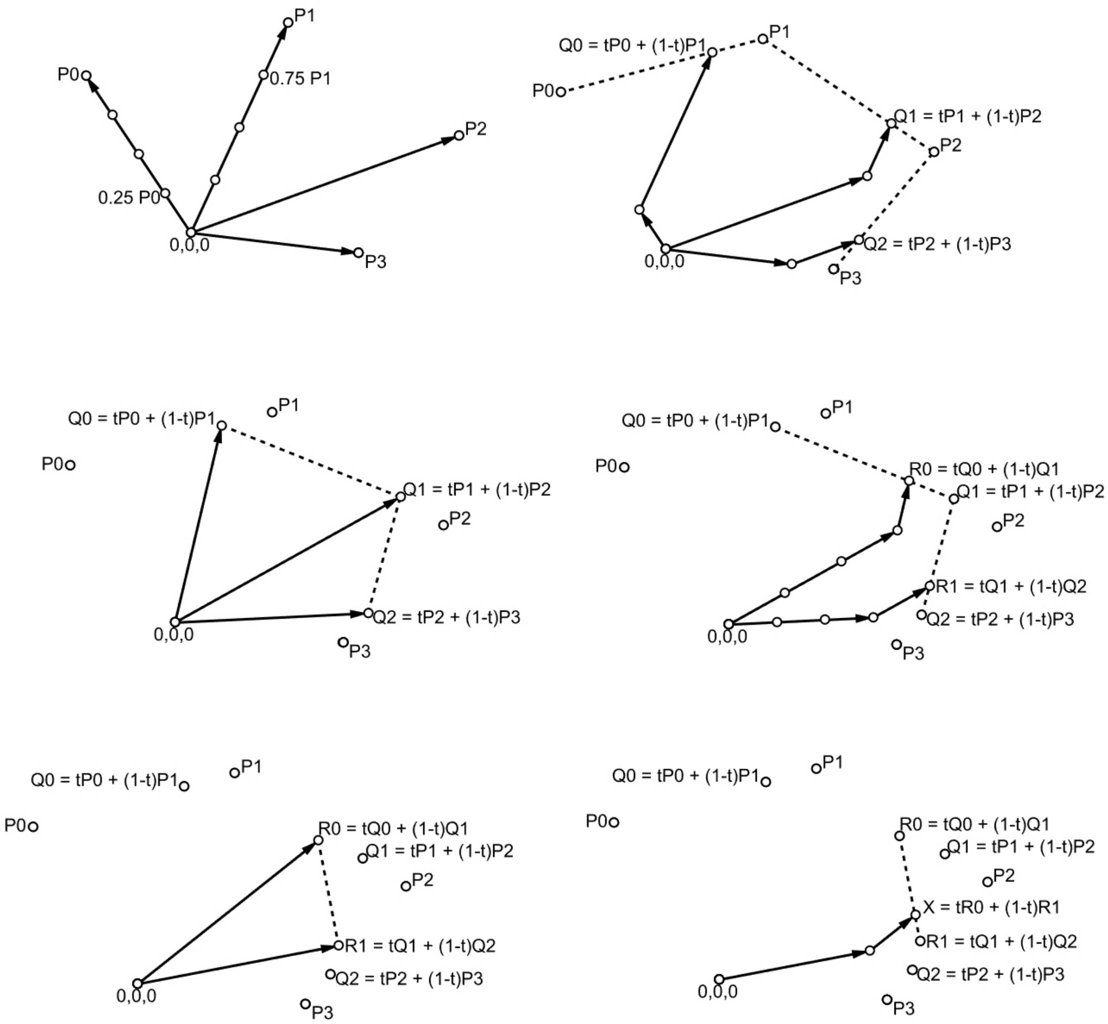

Figure 6: Degree 3 Bezier curve construction

The general formula for the n-degree Bezier curve can be found using Bernstein polynomials and Binomial functions;

$$\sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i P_i$$

$$\binom{n}{i} = \frac{n!}{i!\,(n-i)!}$$

Guitar Surface

Figure 7: The guitar that is going to be measured.

The shape of a guitar is too complex to be defined by four control points. Therefore, the shape is divided into 3 sections on both sides. Using multiple pieces to find a function with the Beizer application is called a piecewise function. The below figure illustrates this approach. After measuring with a ruler meter, the length of the guitar body was found to be 47 ± 0.1 cm. Therefore, the highest point value will be given as 47 to keep it to scale.

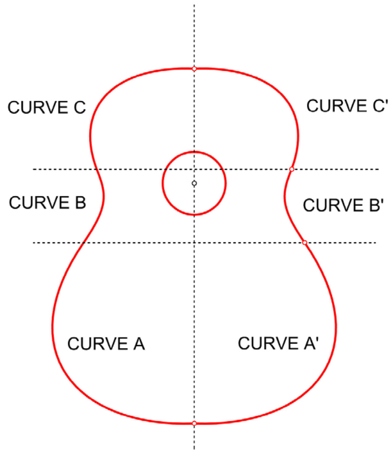

Figure 8: Division of the guitar shape into six sections.

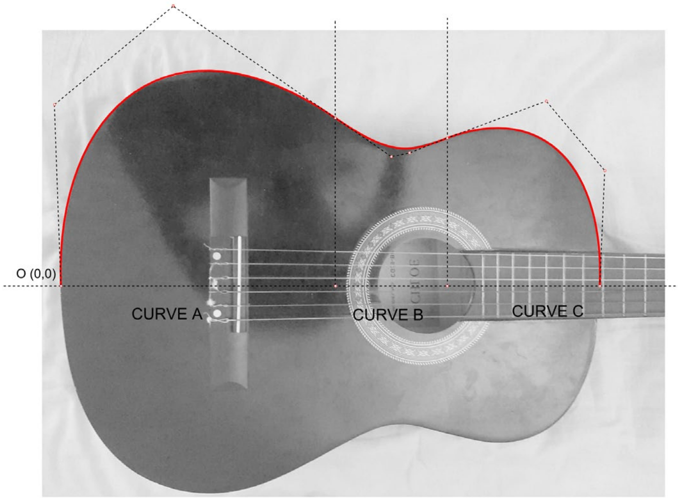

Figure 9: Drawing the three Bezier curves (degree-3) by approximating the guitar shape.

Each of the three sections contains a degree-three Bezier curve so that they each have four control points. The figure above shows the approximation of these points on a photo of the guitar. To make the curves continuous while piecewise, the first and the last control points of the curves are overlapping. For example, A_4 and B_1 are overlapping. To make the overall shape smooth the first and last control points of the curves are made collinear. For example, A_3 $A_4(B_1)$ and B_2 are collinear. Finally, the drawing is made by overlapping the symmetry axis of the guitar to the x-axis. So, the area between the sections and the x-axis would give the area of that section on the guitar.

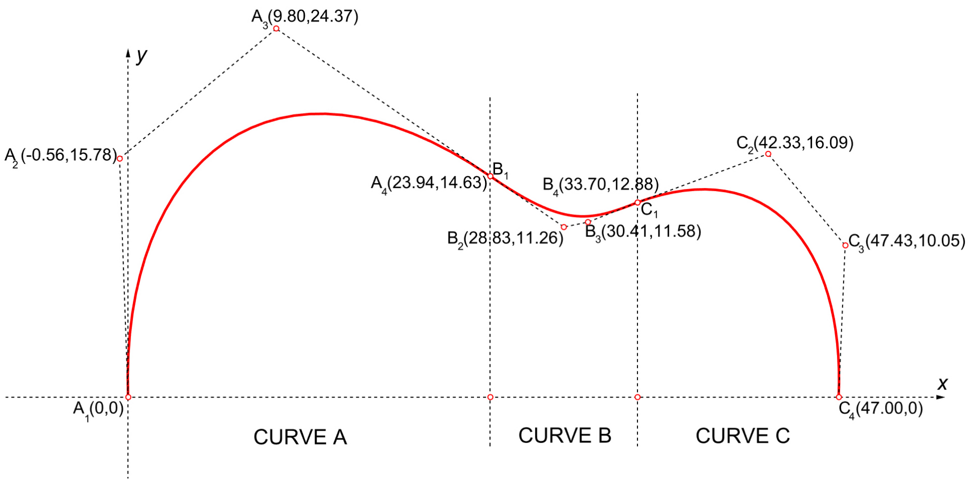

Figure 10: Coordinates of the control points.

Appling the Bezier Formula

The third degree Bezier formula (explained in earlier chapters) will be used to determine the parametric equations of the three sections:

$$X = (1-t)^3 P_0 + 3(t-1)^2 t P_1 + 3(1-t) t^2 P_2 + t^3 P_3$$

Troughout this study the CAD drawigns are made by Rhinoceros software which can be used to draw Bezier curves of any degree. Also, the area between the curves and the x axis can be calculated to check the results of the area calculation. This software is an industry standard for architechs and disigners. Moreover, the formulas obtained in this study are also checked with the software to validate the parametric curves.

Curve A

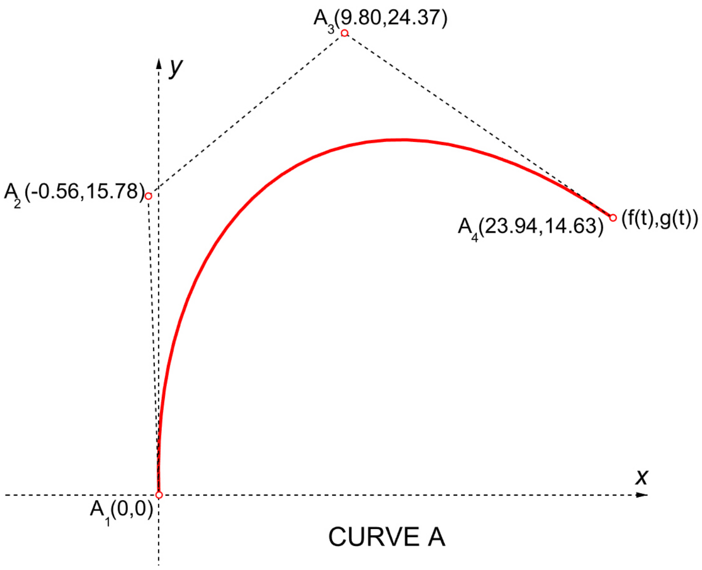

Figure 11: Control points of Curve A section

The third-degree Bezier curve formula will be applied to the Curve A section control points. Those points and their values are found using the Rhinocer CAD application. Also, in CAD a value of t (0.5) is given to visualize geometrical application proof. As Curve A starts from the origin (0,0), interpolation between A_1 and A_2 points will give a relatively short equation to later sections.

$$\mathrm{f(t)} = (1-\mathrm{t})^3 * 0.0 + 3 * (1-\mathrm{t})^2 * \mathrm{t} * -0.56 + 3 * (1-\mathrm{t}) * \mathrm{t}^2 * 9.80 + \mathrm{t}^3 * 23.94$$

$$g(t) = (1-t)^3 * 0.0 + 3 * (1-t)^2 * t * 15.78 + 3 * (1-t) * t^2 * 24.37 + t^3 * 14.63$$

The equations for Curve A can be shorted as;

$$\mathrm{f(t)} = 23.94\mathrm{t}^3 - 1.68\mathrm{t}(1-\mathrm{t})^2 + 29.4(1-\mathrm{t})\mathrm{t}^2$$

$$\mathrm{g(t)} = 14.63\mathrm{t}^3 + 47.34\mathrm{t}(1-\mathrm{t})^2 + 73.11(1-\mathrm{t})\mathrm{t}^2$$

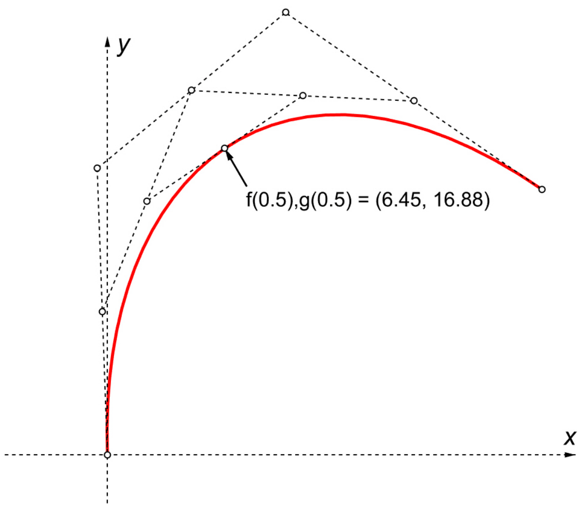

Figure 12: An example calculation to find t=0.5 on Curve A

To test the cubic formulas t (0.5) is evaluated and shown visually above. If the point from any t value is correct it should be on the curve, therefore giving t half value would give parametrically the middle point of the curve. Below is the algebraic computation of the point on Curve A at the parametric middle point to 2 decimal points.

$$f(0.5) = 23.94 * 0.5^3 - 1.68 * 0.5 * (0.5)^2 + 29.4 * (0.5) * 0.5^2 = 6.45$$

$$g(0.5) = 14.63 * 0.5^3 + 47.34 * 0.5 * (0.5)^2 + 73.11 * (0.5) * 0.5^2 = 16.88$$

Curve B

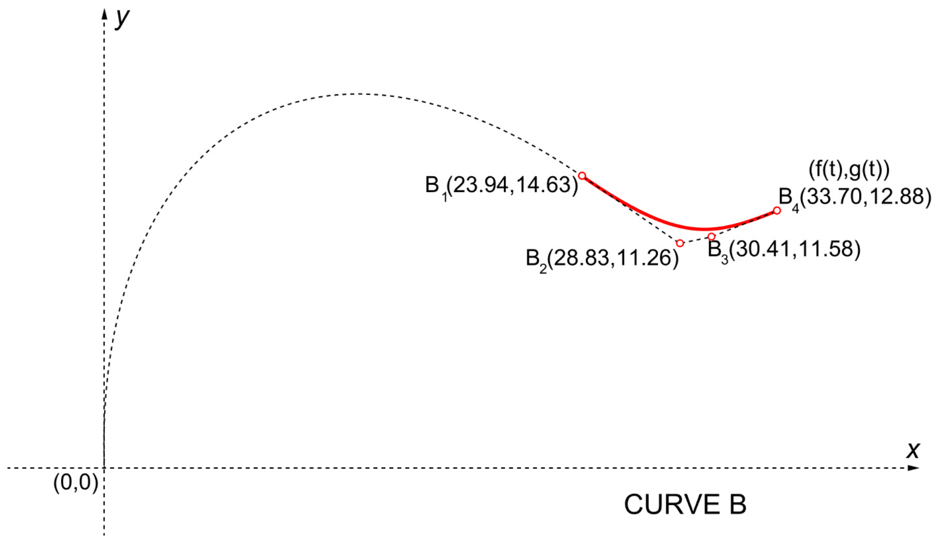

Figure 13: Control points of Curve B section

The same process will be followed in Curve B and C sections. That is calculating the cubic formula with the t parameter and giving t (0.5) points and labeling points visually. From the third-degree formula, a longer equation from the previous section will be achieved as expected. The point values were found with the same command before and ending points are accurately connected.

$$f(t) = (1-t)^3 * 23.94 + 3 * (1-t)^2 * t * 28.83 + 3 * (1-t) * t^2 * 30.41 + t^3 * 33.70$$

$$g(t) = (1-t)^3 * 14.63 + 3 * (1-t)^2 * t * 11.26 + 3 * (1-t) * t^2 * 11.58 + t^3 * 12.88$$

The equations for Curve B can be shorted as;

$$f(t) = 33.70t^3 + (1-t)^3 23.94 + 86.49(1-t)^2 + 91.23(1-t)t^2$$

$$g(t) = 12.88t^3 + (1-t)^3 14.63 + 33.78(1-t)^2 t + 34.74(1-t)t^2$$

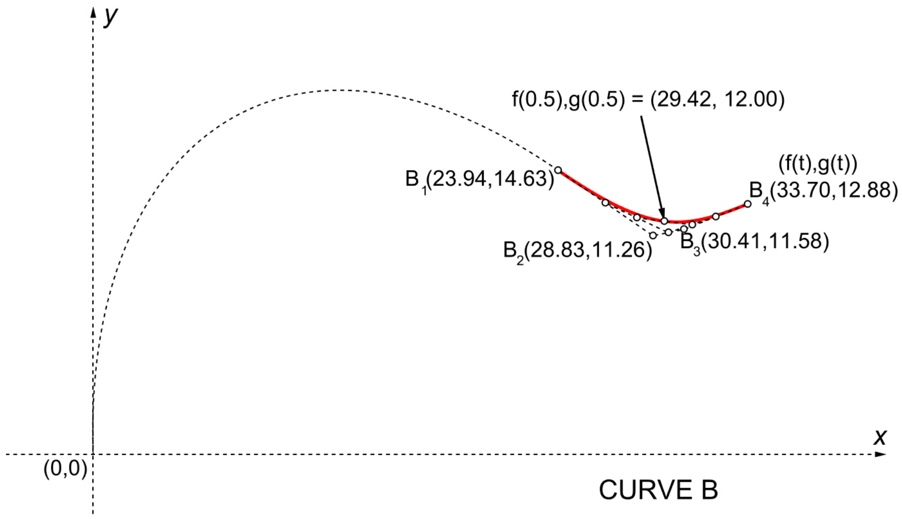

Figure 14: An example calculation to find t=0.5 on Curve B

$$f(0.5) = 33.70(0.5)^3 + (0.5)^3 23.94 + 86.49(0.5)^2 + 91.23(0.5)0.5^2 = 29.42$$

$$g(0.5) = 12.88(0.5)^3 + (0.5)^3 14.63 + 33.78(0.5)^2 t + 34.74(0.5)0.5^2 = 12.00$$

Curve C

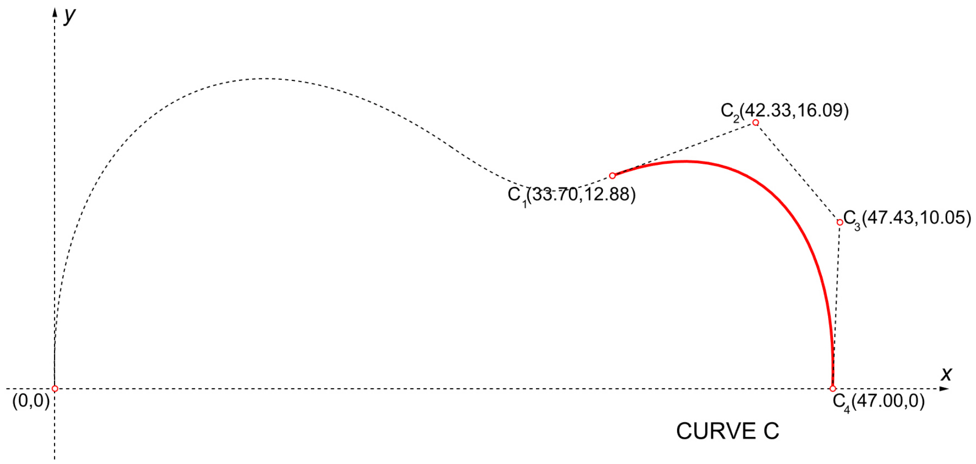

Figure 15: Control points of Curve C section

In the last section same steps were followed and achieved these results.

$$f(t) = (1-t)^3 * 33.70 + 3 * (1-t)^2 * t * 42.33 + 3 * (1-t) * t^2 * 47.43 + t^3 * 47.0$$

$$g(t) = (1-t)^3 * 12.88 + 3 * (1-t)^2 * t * 16.09 + 3 * (1-t) * t^2 * 10.05 + t^3 * 0.0$$

The equations for Curve C can be shorted as;

$$f(t) = 47.00t^3 + 33.70(1-t)^3 + 126.99(1-t)^2 t + 142.29(1-t)t^2$$

$$g(t) = 12.88(1-t)^3 + 48.27(1-t)^2 t + 30.15(1-t)t^2$$

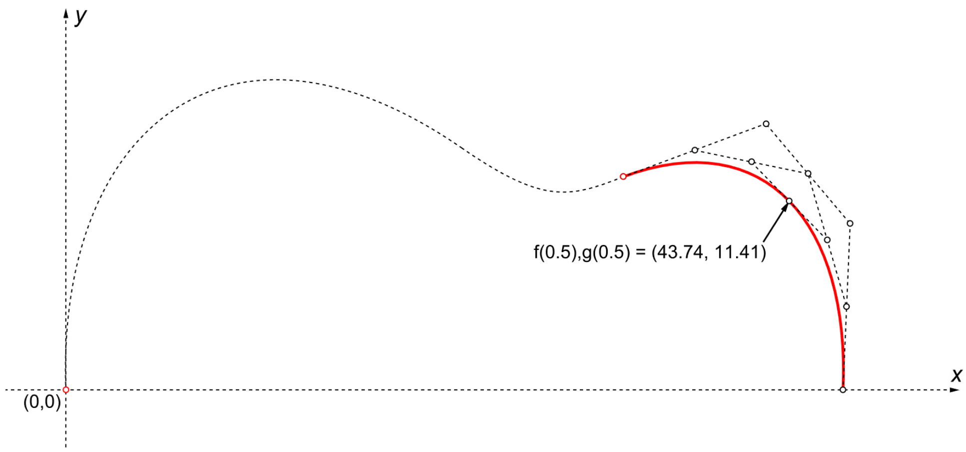

Figure 16: An example calculation to find t=0.5 on Curve C

Lastly, to visually demonstrate the equation and geometrical connectivity t will be given as 0.5 for the last section.

$$\mathrm{f}(0.5) = 47.00(0.5)^3 + 33.70(0.5)^3 + 126.99(0.5)^2 0.5 + 142.29(0.5)0.5^2$$

$$\mathrm{g}(0.5) = 12.88(0.5)^3 + (0.5)^3 + 48.27(0.5)^2 0.5 + 30.15(0.5)0.5^2$$

Determining Surface Area

To calculate the surface area of the guitar, applying the integral rule to the parametric function found with Bezier curves on the 3 sections separately is needed. The three sections are drawn by overlapping the symmetry axis of the guitar on the x-axis. Thus, the area between the sections and the x-axis would give the half of the surface area of that section on the guitar. Using the below equation would give (A) Area enclosed by the parametric function. The result is calculated using an online integral calculator (https://www.integral-calculator.com/) for each section and is validated on the CAD tool's "area" command. The result data is given in cm form with 2 decimal points.

$$A = \int_0^1 g(t)\, f'(t)\, dt$$

For Curve A;

$$f'(t) = -\frac{1071\mathrm{t}^2 - 3276\mathrm{t} + 84}{50}$$

$$A = \int_0^1 \frac{(14.63t^3 + 47.34t \cdot (1-t)^2 + 73.11(1-t)t^2)(-1071t2 + 3276t - 84)}{50}\, dt = 390.28$$

For Curve B;

$$f'(t) = -\frac{24441t^2 - 49908t + 24480}{100}$$

$$A = \int_0^1 \frac{(12.88\mathrm{t}^3 + (1-\mathrm{t})^3 14.63 + 33.78(1-\mathrm{t})^2\mathrm{t} + 34.74(1-\mathrm{t})\mathrm{t}^2)\,(-24441t^2 + 49908t - 24480)}{100}\, dt = 123.95$$

Lastly for Curve C;

$$f'(t) = -\frac{810t^2 + 1978t - 2589}{100}$$

$$A = \int_0^1 \frac{(12.88t^3 + (1-t)^3 + 48.27(1-t)^2t + 30.15(1-t)t^2)\,(-810t^2 - 1978t + 2589)}{100}\,dt = 160.85$$

So, the total area of the three sections was found 390.28 + 123.95 + 160.85 = 675.08 cm^2. However, this area is only one-half of the surface with a sound hole in it too. After doubling one half 2 * 675.08 = 1350.16 cm^2 and measuring the diameter (8.3cm) of the hole to find the area of the circle (54.44 cm^2), the total area of the guitar surface was found;

$$1350.16 - 54.44 = 1295.72\ cm^2$$

Conclusion

In this paper, Bezier curves were utilized to analyze the shape and the surface area of the guitar. The construction of Bezier curves is based on a series of linear interpolations. This gives the utility to express any Bezier curve in terms of parametric polynomials. This study demonstrated that three third-degree curves are sufficient to approximate the curved shape of a guitar body. The functions derived from this approximation give us the fundamental algebraic definition. Thus one can draw the shape with the help of these functions. The inspiring part of this study is that the mathematical explanation of curve shapes is strongly related to the developments in technology and engineering. This eliminates the need for physical templates or blueprints which can be worn out. Instead, the algebraic definitions are useful to store any given shape mathematically so that they can be regenerated without any loss of information. Finally, the surface areas between these curves and the x-axis can be calculated.

The Bezier curves are an efficient way to create these algebraic definitions. They are widely used in computer-aided design because a designer can draw and transform any curve easily and visually by placing control points. The designers don't have to know the mathematics behind these curves focusing on their work. However, in this study, these mathematical underpinnings are explained and used to express this hidden relationship between mathematics and design.

References:

- Yazar, T. Compass Construction of Bézier Curves and B-Splines. *Nexus Network Journal* **23**, 789–811 (2021). https://doi.org/10.1007/s00004-020-00542-9
- Farin, Gerald. (2002). A History of Curves and Surfaces in CAGD. Computer Aided Geometric Design - CAGD. 10.1016/B978-044451104-1/50002-2.
- *On The Spline: A Brief History of the Computational Curve (Full) – Alastair Townsend*. alatown.com/spline-history-architecture.
- *Chapter 4: B Zier Curves Cubic and beyond | GlobalSpec*, www.globalspec.com/reference/60999/203279/chapter-4-b-zier-curves-cubic-and-beyond [accessed 17 Dec, 2023]
- A Study of Curved Boundary Representations for 2D High Order Euler Solvers - Scientific Figure on ResearchGate. Available from: https://www.researchgate.net/figure/Bezier-curves-of-various degrees_fig1_220395548 [accessed 17 Dec, 2023]
- https://www.integral-calculator.com/ [accessed 17 Dec 2023]