THE RELATIONSHIP BETWEEN GAUSSIAN CURVATURE AND ARCHITECTURAL SURFACE PANELING

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Abstract. Since the design of free-form architectural surfaces becomes easier, questioning and foreseeing the feasibility of the construction of these surfaces became important. Such an inquiry requires sufficient knowledge of architectural geometry besides the knowledge of materials and structural systems. In this article, a preliminary example of a guide that supports the design and production process of building surfaces with different geometric properties is presented. This guide aims to reveal the relationship between Gaussian curvature which is an intrinsic geometric feature of architectural surfaces, and some of the widespread paneling strategies. After the literature review and the description and classification of curvature and paneling concepts via architectural examples, a comparative table has been created. The resulting table facilitates answering the question of whether it is possible geometrically to build architectural surfaces with different paneling strategies, especially in the early phases of architectural design. Thus, this preliminary information can be helpful to the designer regarding the estimated cost, the materials, and the technologies to be selected. Further development of such guides will help architects and students to use geometry more consciously.

Keywords. Paneling; architectural geometry; Gaussian curvature; surface geometry.

INTRODUCTION

Paneling free-form building surfaces is an interdisciplinary field of application and research. This area encompasses components from design and engineering, as well as geometry, mathematics, and computer science. In recent architectural discourse, two primary sources stand out when exploring this subject. In his 2006 publication "Algorithmic Architecture", Kostas Terzidis argues that algorithms are not just a series of computer codes or a mechanical expression language used for step-by-step problem-solving, but also an ontological structure with deep philosophical, social, and artistic impacts (Terzidis, 2006). This concept, as articulated by Terzidis, has brought forth a range of ideas suggesting that architects and computer programmers share some common ground in their expertise. Coding has evolved from being a luxury for designers to becoming a new and effective tool in their toolkit. A year later, mathematician Helmut Pottmann, along with geometry experts and designers, compiled the book "Architectural Geometry," reminding architects that geometry is not just a resource to be consumed, but can still be a subject of research (Pottmann et al., 2007). The parametric design tools developed by the generation that graduated from architecture schools during this period are worth examining as synthesizers of these ideas. These tools have led to the creation of general-purpose generative design software, establishing a shared educational system in virtual environments. Notable examples of these tools include David Rutten’s "Grasshopper" developed after graduating from Delft University of Technology in 2006, as well as Daniel Piker’s "Kangaroo," Giulio Piacentino’s "Weaverbird," Mateusz Zwierzycki’s "Anemone," Mostapha Sadeghipour Roudsari’s "Ladybug, Honeybee, Butterfly," and Milos Dimcic’s "eVe."

These tools and programming languages, developed by architects, help utilize time and information efficiently in architectural practices and are also recognized as laboratory tools in academic studies. This system of thought and the background of these tools prompt designers to revisit geometry and algorithms from a new perspective. Consequently, computational geometry is becoming a useful body of knowledge for architects. The convergence of architecture, mathematics, and computer science (Figure 1) fosters the emergence of research topics with easily foreseeable economic value and appeal. "Paneling of architectural surfaces" is a contemporary research topic that emerged from this intersection.

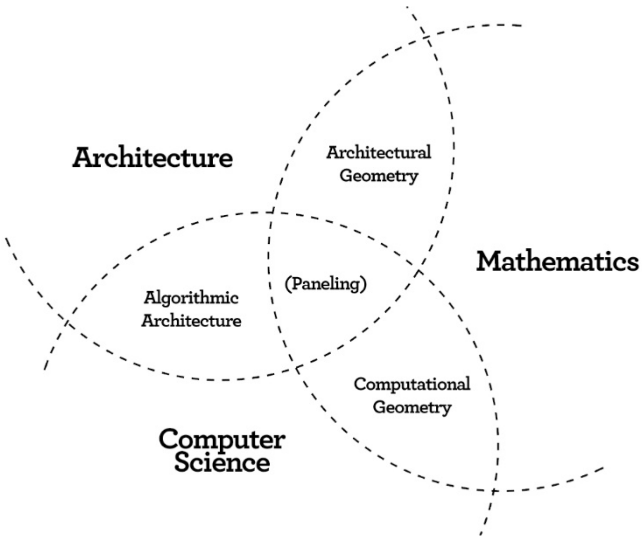

Figure 1. The context of the research topic

Architectural paneling involves dividing building surfaces into segments that meet criteria such as cost, production capabilities, construction materials, energy conservation, etc. Architects often require solutions to complex, multi-variable problems of surface paneling, prompting collaboration with experts from various disciplines. Mathematicians and computational geometry specialists contribute to the basis of these problems by developing surface discretization algorithms or adapting existing ones to specified problems. Computer programmers then translate this basis into tools that compute and propose solutions tailored to the design problem. The solutions obtained range from simple scripts addressing the specific needs of individual projects to more general-purpose extensions, plugins, and software. Given the diversity of architectural problems, a universal paneling tool that can offer the optimal solution in every scenario is not feasible. From an architectural design perspective, abstract surfaces created in early design stages can be paneled during later phases, or they can initially be designed to fit a paneling method on purpose. Architectural paneling methods vary widely, from simple solutions achievable with traditional drawing tools to those requiring high computational costs and thus necessitating specialized software.

Accessing advanced paneling solutions, particularly in cases where specialized work groups, consultations, or software are unavailable, poses challenges in achieving the efficiency and economic benefits promised by these solutions. Additionally, some popular solutions, admired for their geometric ingenuity and refined appearances, may lose their relevance and purpose if they neglect geometric achievements and the benefits outlined above. Academic studies in this field increasingly define a specialized expertise. This article proposes a method to expedite the transformation of research findings in the evolving field of architectural paneling into practical knowledge. The introductory section of the article provides a general overview of existing research on this topic (as of 2018). Analyses related to architectural paneling primarily depend on a good understanding of surface geometry, a mathematical component of the subject. This article focuses on surface curvature, an intrinsic geometric property of architectural surfaces. Subsequent sections examine various surface paneling approaches, selected for their simplicity using basic architectural drawing knowledge and contemporary design tools. These preliminary studies culminate in a comparison that explores the feasibility of applying different paneling approaches to surfaces with varying curvatures. This comparison aims to facilitate architects in forming anticipations based on fundamental surface geometry knowledge, thus preventing or mitigating the dulling of such insights before the need for specialized software or consultations arises. The final section discusses further research progress.

LITERATURE REVIEW

Research focusing on the paneling of architectural surfaces has significantly expanded and gained momentum over the past decade. Studies in this area address fundamental research problems such as ensuring that surface panels are planar, accurately following surface curvatures, or minimizing the variety of panels to reduce costs. One common research topic is calculating how free-form surfaces can be constructed using panels cut from planar materials. While triangulated surfaces are a widely used solution, it is known to be more costly compared to quadrilateral paneling in the context of architectural surface paneling, due to the greater number and variety of seam details triangles produce compared to quadrilaterals (Pottman et al., 2007, Pottmann et.al, 2007b). The convergence of more than three panels at the same point begins to complicate static calculations. Triangulation often results in scenarios where six panels converge at the same point at different angles, making it difficult to compute points without torsion (Cutler and Whiting, 2007, Pottman et.al. 2007b).

Another current research topic is dividing surfaces using planar quadrilaterals. Berk's doctoral thesis examines various algorithms that partition surfaces into planar quadrilateral panels, discussing their advantages and disadvantages (Berk, 2012). Some studies suggest that specific mathematical surface constructions are more suitable for planar quadrilateral paneling (Glymph et.al, 2004, Berk, 2012, Pottmann, 2007) when deciding on quadrilateral paneling before commencing design. Conversely, several researchers explore the post-design generation of planar quadrilateral panels for free-form surfaces, aiming to minimize deviation from the surface (Pottmann, 2007). For instance, any surface can utilize principal curvature lines for quadrilateral planar paneling (Liu et.al., 2011).

New methods are being developed to produce panels with more edges than quadrilaterals. For example, Pottmann et al. highlight the close relationship between surface curvature and paneling with hexagonal panels tailored to fit the surface curvature (Pottmann et.al., 2014). Rörig et al. (2014) seek solutions for placing hexagonal panels optimally on surfaces, emphasizing criteria such as flatness, equilateral sides, and uniformity (Rörig, et. al. 2014). Research also examines manufacturing opportunities related to surface curvature rather than the number of panel edges. For example, the Landesgartenschau exhibition hall, developed and built in 2014 by Achim Menges and colleagues, exemplifies paneling that conforms to Gaussian curvature (Menges, et.al., 2014). Another study by Pottmann et al. (2008) examines the use of single-curvature panels instead of planar panels, while Berk's doctoral thesis explores the potential use of non-planar panels on such surfaces (Berk, 2012).

Eigensatz et al. (2010) present a cost-focused study for paneling free-form surfaces using molds of various types like plane, cylindrical, paraboloid, cubic, or toroidal, highlighting cost advantages. Rather than focusing on specific geometric problems, another research perspective provides a broader view, noting a shift in the construction industry from "Can we build this?" to "Should we build this?" after comparing various paneling approaches, underscoring the importance of healthy communication among architects, engineers, and contractors (Hambleton et.al., 2009).

Research in this field is supported by research and development (R&D) groups such as Evolute (www.evolute.com), Gehry Technologies (www.gehrytech.com), and Mesh (www.meshconsultants.ca), which provide consultancy to architectural offices. These collaborations involve architects, structural engineers, mathematicians, and computational design experts working together to develop solutions that effectively panel complex surfaces according to criteria like cost, material, and production capabilities. Some research in this field focuses on transferring architectural application experiences (Schifner et.al., 2012). For example, Kaijima and Michalatos (2007) discuss a project where different surface segmentation approaches were experimented with, examining design criteria and segmentation options. Studies also show that software plugins are being used as auxiliary tools, where these tools are tested and compared. Henriksson and Hult's master's thesis (2015) examines current design tools that can optimize load distributions and panel similarities during the segmentation of free-form surfaces.

The "Advances in Architectural Geometry" symposiums serve as platforms where research on architectural surface paneling is presented and published, featuring papers collaboratively authored by researchers from various disciplines. Software plugins mentioned earlier are actively used by companies specializing in designing and producing precast building facades. However, the transformation of this experience into R&D in the industry and academic responses capable of meeting this demand in architectural education are still in their infancy.

CLASSIFICATION OF ARCHITECTURAL SURFACES

Research on surface curvature has evolved through various stages and definitions since ancient Greece, gaining momentum with the development of Calculus by Newton and Leibniz in the 17th century following works by Descartes, Kepler, Fermat, and Huygens (Stillwell, 2010). The concept of curvature derived from these developments has led to significant advances in fields such as mathematics and physics, prompting a reassessment of propositions in Euclidean geometry, as well as clarifying the mathematical and physical meanings of space, time, and gravity (Stillwell, 2010). Today, the curvature of curves and surfaces falls under the realm of differential geometry. This article focuses on surface curvature as relevant to its application in architecture, specifically in terms of intrinsic curvature, which refers to the curvature inherently possessed by objects independent of the space they occupy. The method for calculating the intrinsic curvature of surfaces was defined by Carl Gauss in the 19th century and is therefore named Gaussian curvature. To explain Gauss's method for calculating surface curvature, it is first necessary to understand how the curvature of curves is calculated. According to this definition, curvature (K) is inversely proportional to the size of the osculating circle at a point t on a curve AB (Figure 2). This circle is not just any tangent circle but the one that best approximates the curve. The center of this circle is called the curvature center. As curvature increases at point t, the radius (r) of the osculating circle decreases. This indicates that the curve is more curved at that point (K increases). As the curve flattens at point t, the osculating circle also grows (r increases). The axis connecting the center of the osculating circle with point t is perpendicular to the curve.

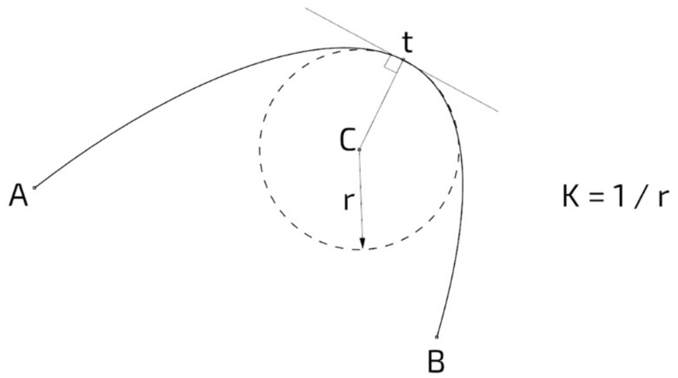

Figure 2. Calculation of curvature via osculating circle

The surface curvatures studied in this research are measured using two osculating circles created using the method described above, taken in both directions from a point t on the surface (Figure 3). Principal curvatures in both directions (K1 and K2) are obtained through the osculating circles of these sections passing through point t. These two sections are always perpendicular to each other (Stillwell, 2010).

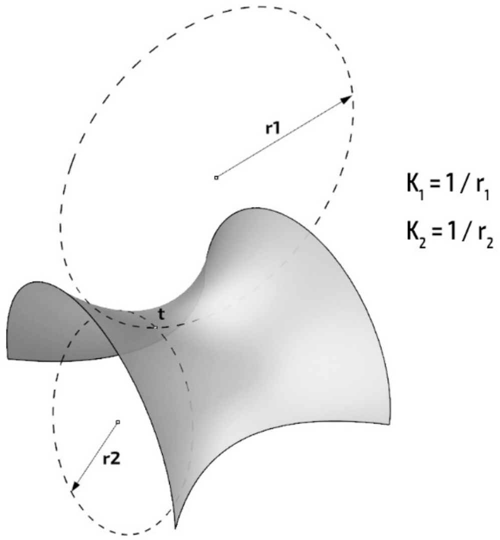

Figure 3. Calculation of surface curvature

After obtaining the two principal curvature values, the curvature at point t on the surface can be calculated in the following forms:

(K1 + K2) / 2: The arithmetic mean of the principal curvatures will give the **mean curvature** of the surface at that point.

K1 × K2: The product of the principal curvatures will give the **Gaussian curvature** of the surface at that point.

Since mean curvature is a ratio, it is an independent value from the size of the measured surface. Therefore, it is not common in architectural surfaces and is generally used in other fields of computational geometry. The most common use of mean curvature in architecture is in minimal surfaces. The mean curvature on minimal surfaces is zero at every point (K1 + K2 = 0). This is possible in surfaces where the principal curvatures in both directions are continuously equal and opposite. One of the most well-known examples demonstrating the scale independence of mean curvature is the experiments Frei Otto conducted with soap bubbles to create minimal surfaces. These experiments made it possible to design and manufacture the roof of the 1972 Munich Olympic Stadium.

On the other hand, the Gaussian curvature addressed in this study is related to the dimensions of the surface being studied. As the surface increases in size, Gaussian curvature decreases. Therefore, it encapsulates information that can be measured and scaled. Performance aspects linked closely with Gaussian curvature include a surface's ability to unfold on a flat plane. If at least one of the osculating circles is infinitely large, the section taken is a straight line, indicating zero Gaussian curvature at that point (Ka × 0 = 0). Surfaces with zero Gaussian curvature at every point can be unfolded into a flat plane or constructed from flat materials by folding (Figure 4, left). Folding motions that do not involve stretching or shrinking also maintain Gaussian curvature (Conway et al., 2010). Therefore, surfaces with zero Gaussian curvature can be matched isometrically to planes and are thus developable. This knowledge is applied in computer graphics and architectural modeling to apply texture to objects for obtaining photorealistic images.

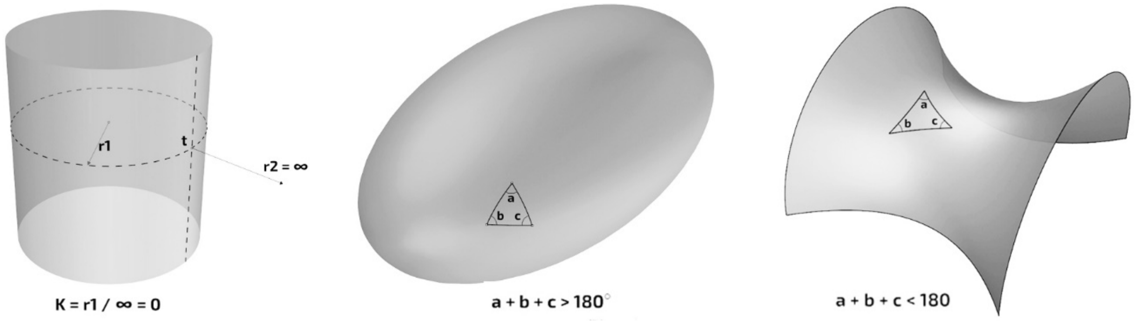

Figure 4. Gaussian curvature; (left) zero, (middle) positive, (right) negative curvature cases

Surfaces with non-zero curvature, hence unable to be unfolded into a plane, are known in architectural discourse as doubly curved. Non-zero Gaussian curvatures are either positive or negative. Positive Gaussian curvature exists where both osculating circles lie on the same side of the surface (Figure 4, middle). In regions with positive Gaussian curvature, the edges of the surface curve towards the same side, indicating a tendency for closure. On surfaces with positive Gaussian curvature, the sum of interior angles of a triangle drawn on the surface exceeds 180 degrees, and the circumferences of circles drawn on the surface are greater than 2 × Pi × R. Points on a surface with positive Gaussian curvature are termed elliptic points, and surfaces with positive Gaussian curvature are called synclastic surfaces. Surfaces defining closed volumes typically have positive Gaussian curvature. If a surface has constant positive Gaussian curvature at every point, it is part of a sphere or a spherical segment. Shapes containing positive curvature generally show a tendency to close and typically form structures that perform under pressure distribution. The geometric solution of the sphere designed by Buckminster Fuller in 1982 Spaceship Earth in Florida, USA, illustrates positive curvature. Geodesic sphere design is closely linked to constant positive Gaussian curvature cases. It is possible to state that designs typically referred to as thin shell structures in architectural discourse generally contain surfaces with constant or variable positive Gaussian curvature. An example of surfaces with variable positive Gaussian curvature in architecture is the St. Mary Axe building in London, designed by Foster + Partners and completed in 2004. The shell of the building is a surface with increasing positive curvature as it approaches the peak (Figure 5).

Figure 5. St. Mary Axe Building (London, England), Architect: Foster + Partners, 2004

If the two osculating circles are on opposite sides of the surface, the Gaussian curvature of the surface at that point is negative (Figure 4, right). In regions with negative Gaussian curvature, the surface's two directions curve in opposite directions, indicating a tendency not to close at that point. On surfaces with negative Gaussian curvature, the sum of interior angles of a triangle drawn on the surface is less than 180 degrees, and the circumferences of circles drawn on the surface are less than 2 × Pi × R. Points on a surface with negative Gaussian curvature are termed hyperbolic points, and surfaces with negative Gaussian curvature are called anticlastic surfaces. If a surface has negative Gaussian curvature at every point, it is not closed. The pseudosphere can be shown as an example of a surface with negative Gaussian curvature at every point. Shapes consisting of surfaces with negative curvature tend to be referred to as canopies in architectural terms due to their characteristics described above. The garage gate built by Santiago Calatrava in Coesfeld, Germany, in 1985, featuring surfaces with variable negative curvature, can be given as an example of mathematical surfaces. As one of the best-known examples of architectural interpretations of hyperboloid surfaces with variable negative curvature, the Brasilia Cathedral, completed by Oscar Niemeyer in Brazil in 1970, illustrates this (Figure 6). The tendency towards infinite opening created by negative curvature was used in the Palmira Chapel in Mexico designed by Félix Candela. Candela designed the thin reinforced concrete roof as a hyperbolic paraboloid, joining researchers like Buckminster Fuller, Frei Otto, Heinz Isler, and other contemporaries in the era where modern architecture intersected with mathematics and geometry. Another surface class widely used in architecture is ruled surfaces. It often misleadingly appears to have straight lines in one direction, suggesting they are surfaces with zero Gaussian curvature and thus can be unfolded flat. However, these surfaces do not have zero principal curvature, and their Gaussian curvature (excluding focal points) is negative everywhere. Ruled surfaces used in the facade design of the Walt Disney Concert

Hall in Los Angeles are therefore supported by flat supports but are not openable in terms of surface paneling (double-curved surfaces) (Figure 7).

Figure 6. Brasilia Cathedral (Brazil), Architect: Oscar Niemeyer, 1970

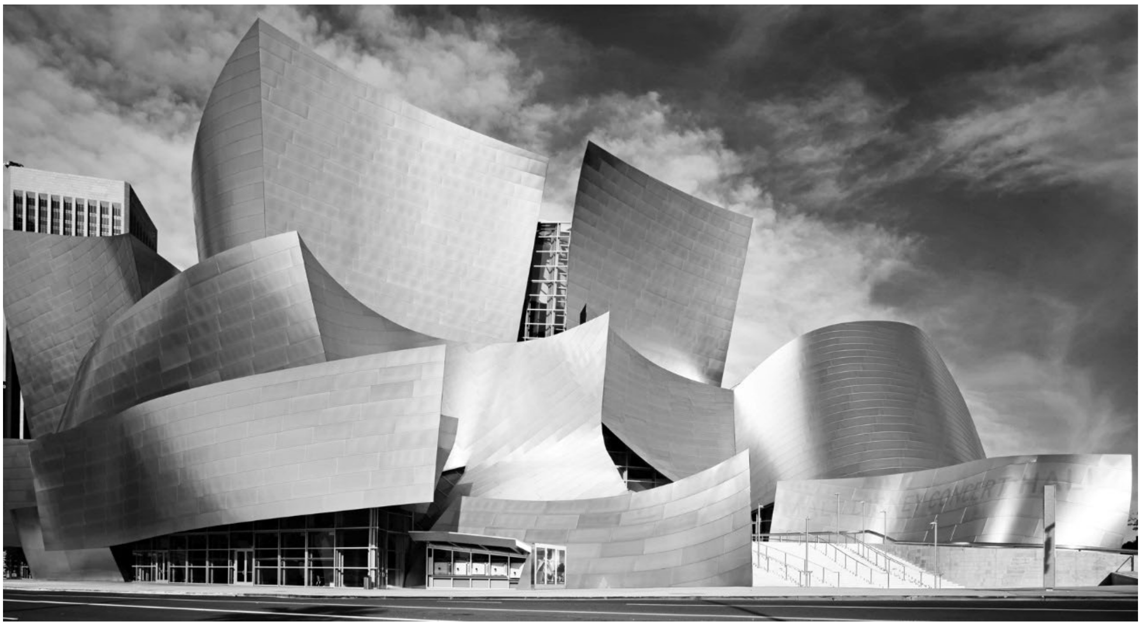

Figure 7. Walt Disney Concert Hall (USA), Architect: Frank Gehry, 2003

Although the fundamental conditions related to Gaussian curvature (regular or variable positive, negative, and zero curvature) described above encompass all possibilities, these three conditions can be mixed in different areas of surfaces for different purposes. These surfaces, defined in architectural jargon as free-form surfaces, can be observed in places like

Zaha Hadid Architects Heydar Aliyev Center completed in Baku, Azerbaijan in 2012, or the canopy of Nordpark Railway Station built in Austria in 2007 (Figure 8). In such facade surfaces, negative, positive, and zero curvatures are all mixed.

Figure 8. Nordpark Railway Station (Austria), Architect: Zaha Hadid Architects, 2007

Another significant example of such surfaces, entering architectural discourse as "Blobitecture," is the Bubble designed by Franken \ Architekten Gmbh and completed in Frankfurt in 1999. The bulging sections around the blobs exhibit positive curvature, while the connecting surfaces formed where two blobs approach each other flex and extend, acquiring negative curvature. Increasing the number of such examples enables the perception of differences in surface curvatures through observation alone, without any calculations. The classification of architectural surfaces according to their Gaussian curvatures is summarized in Table 1.

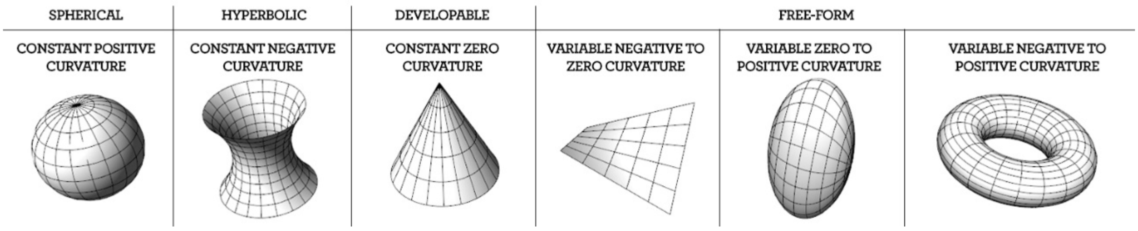

Table 1. Surface classes according to their Gaussian curvatures

ARCHITECTURAL PANELING BASICS

Surface discretization involves defining subsets of mathematical surfaces. The mathematical foundation of architectural paneling lies within the domain of surface discretization. This process typically involves placing reference point sets on parametric surfaces to establish boundaries and define neighborhood relationships among these points (Frank, 2009). Various methods can be used to define point sets and neighborhood relationships, with the choice of method depending on the expected success of the outcome. Within the scope of this study, some of the basic surface discretization approaches are selected. The selection criteria are as follows:

- Examples which are known for their use, particularly in early design stages,

- Approaches suitable for both traditional computer-aided design tools and parametric modeling and scripting,
- Approaches aiming to represent the research area as comprehensively as possible, focusing on diverse criteria for success,
- Approaches capable of adhering to falsifiability principles, due to the rapid advancements and updates in the research field.

Contouring

This paneling approach involves taking sections on the surface, placing points on these sections, and connecting the points to create panels (Figure 9). The result obtained from this approach depends on the direction and spacing of the section planes. The architectural examples paneled with similar approaches include the Kunsthaus completed in Graz, Austria in 2003, designed by Colin Fournier and Peter Cook. The panels of this structure are all different from each other and doubly curved. Foster + Partners' Sage Gateshead building completed in London in 2004 demonstrates a specific case where panels created using this method can be planar (Figure 10). In this example, special surface geometries called translational are paneled with flat rectangles. For this to happen, the source surface must be specifically tailored to fit this paneling approach, and sections must be taken accordingly.

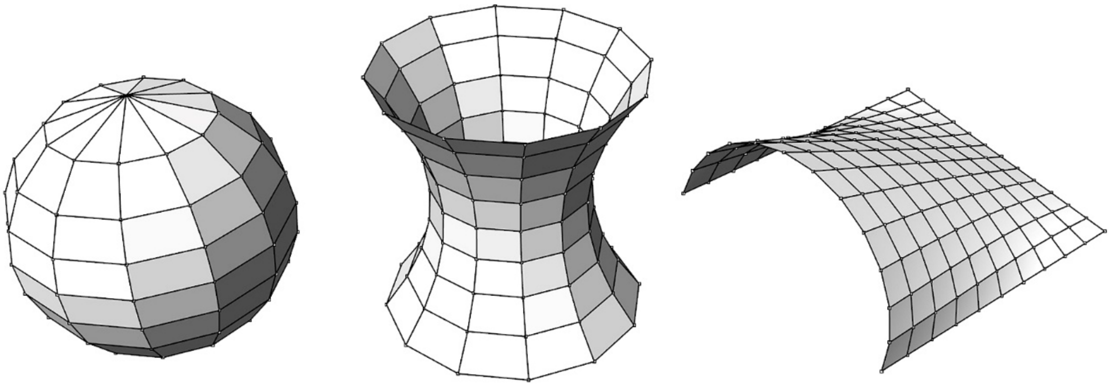

Figure 9. Paneling surfaces using the contouring approach

Figure 10. Sage Gateshead (UK), Architect: Foster + Partners, 2004

This method and these examples are significant in emphasizing the relationship between design, production, and geometry. They show that geometry and paneling achievements can serve as starting points in architectural design, alongside other key parameters such as structural systems and spatial organization. In the context of this research, it is important to examine how similar paneling approaches on different source surfaces lead architects to different design processes. The paneling at Kunsthaus, conforming to the free form of the source surface, was carried out as an expression of the design and technology relationship. In the case of Sage Gateshead, however, the geometry of paneling (being planar, and uniform) has become one of the factors guiding the design decisions.

Euclidean Spheres

Euclidean spheres are a three-dimensional adaptation of Euclidean constructions, where two-dimensional shapes are drawn using a compass and straightedge. It involves intersecting spheres with specified radii along the edges of panels (Figure 11). This method allows for creating points at desired distances on the source surface. The paneling of Soumaya Museum in Mexico City, designed by Fernando Romero is an example of this approach. The report of the facade consultation by Gehry Technologies in 2011 (Figure 12) shows that there were several trials on the panelization of the surface. One such experiment appears to have utilized the Euclidean spheres discussed in this study (Gehry Technologies, 2015). While panels created using the Euclidean spheres can effectively control panel sizes, it has been observed that all panels may not be uniform on surfaces with mixed curvatures.

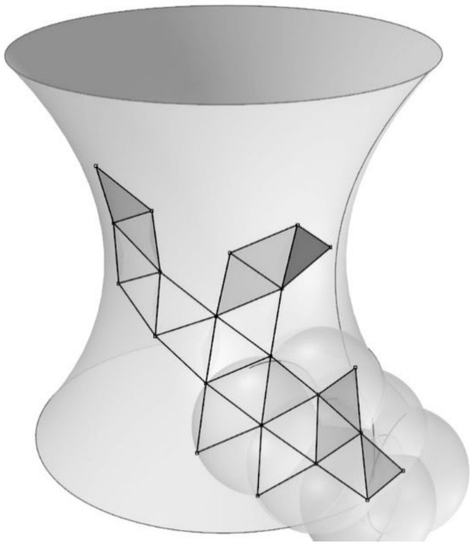

Figure 11. Discretization of a hyperboloid using Euclidean spheres

Figure 12. Soumaya Museum, Fernando Romero, 2011

Quad Extension

In cases where points placed on a surface need to form quadrilaterals, the fourth point is shifted to create planar quadrilaterals (Figure 13). This method, unlike others, creates a different texture by separating from the surface in regions of increased curvature. This method has been applied to many architectural examples. The external facade of The Yas Hotel in Abu Dhabi, completed in 2009 and designed by Asymptote Architecture, was paneled using this approach. Similarly, Foster + Partners utilized this method for the roof design covering the central courtyard of the Smithsonian Institute building completed in 2007 in Washington, USA. The quad extension approach is particularly effective for surfaces with variable curvature and guarantees obtaining planar panels.

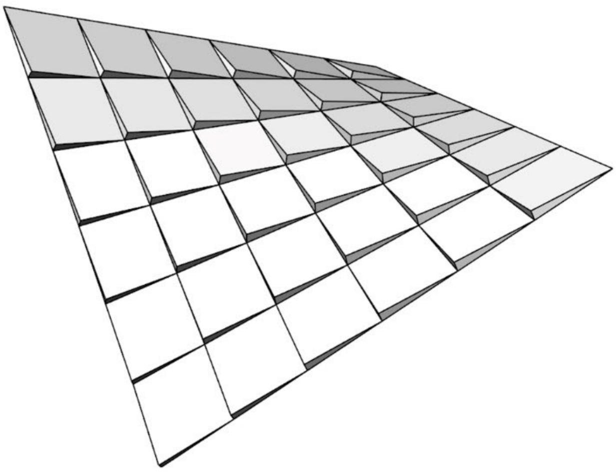

Figure 13. Paneling a hyperbolic paraboloid using the quad extension approach

Marching Cubes

Unlike other methods, the marching cubes approach attempts to approximate surfaces using a cubic reference system (voxel) placed on the surfaces to create units as closely as possible (Figure 14). While these uniform units reduce production costs, they offer the least precise approach to architectural surfaces among the methods presented in this study. Though no directly known building designs use this paneling approach, it can be anticipated that such modular facade designs are frequently used and can be applied to curved surfaces.

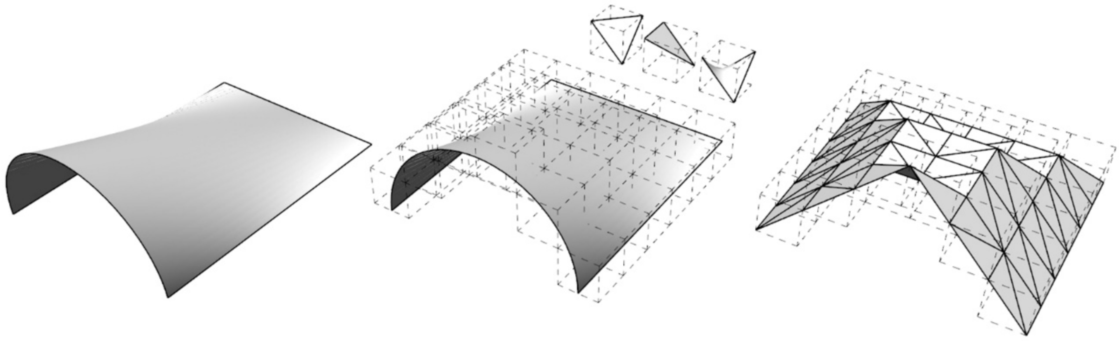

Figure 14. Paneling surfaces using the cube projection approach

Tangent Plane Intersection

The tangent plane intersection approach involves finding tangent planes at points placed on a surface and intersecting these planes to achieve irregular and planar paneling (Figure 15). This paneling approach is the subject of current computational design research, with various experiments conducted. This approach guarantees planar panels and generates diverse polygonal panels, closely related to surface curvature. Effective on surfaces with variable curvature, this method can be successfully applied as the variability in positive or negative curvature decreases. For example, the Trada Pavilion in England, designed and produced by the Ramboll Computational Design group in 2012, is mainly an example of this paneling approach applied to surfaces with positive curvature. A similar approach is seen at the Landesgartenschau Exhibition Hall designed at the Stuttgart Computational Design Institute (ICD) by an interdisciplinary team, offering an innovative solution to how tangent plane intersection approach can adapt to surfaces with variable curvatures.

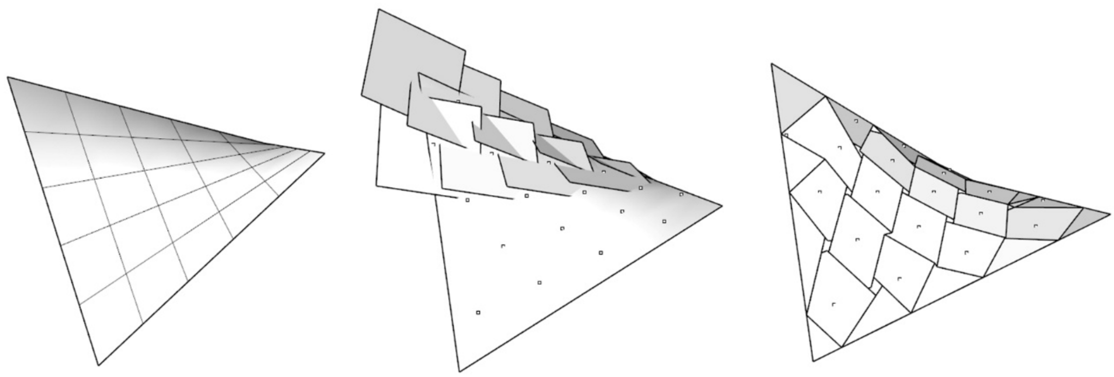

Figure 15. The paneling of surfaces using the tangent plane intersection approach

EXPERIMENTS

The above sections outline the scope of surface curvature and architectural paneling to be addressed in this study. The propositions to be tested are as follows:

- Surface areas based on their curvature (Set A):
  - Regular positive curvature
  - Regular negative curvature
  - Regular zero curvature
  - Variable curvature between negative and zero
  - Variable curvature between positive and zero
  - Variable curvature between negative and positive
- Architectural paneling approaches (Set B):
  - Contouring
  - Euclidean Spheres
  - Rectangular Extrusion
  - Cube Marching
  - Tangent Plane Intersection
- Expected achievements (Set C):
  - All or part of the panels are planar (non-triangular panels)
  - All or part of the panels are regular (equilateral)
  - All or part of the panels are identical

Propositions will be derived using the above sets as follows: For example, "When paneling approach B is used on a surface with curvature type A, achievement C is valid." To test the propositions, paneling approaches were applied to various surface types using drawing, modeling, and coding tools. The selected surface types exemplify the curvature classes mentioned in the propositions (sphere, hyperbolic paraboloid, hyperboloid, ellipsoid, cylinder, etc.). Following the drawings, panels created on these surfaces were analyzed for flatness, equality, or equilateral characteristics. The analysis resulted in conclusions regarding the selected paneling approaches. Instead of seeking generalizations that prove the results are correct under all conditions, the aim was to approach the closest possible truth based on the available data. Particularly, making generalizations about paneling methods on free-form surfaces with variable curvature between negative and positive poses challenges. Hence, the principle of falsifiability was adopted. The falsifiability of comparisons depends on demonstrating that a conclusion reached in one proposition is not valid for another surface with the same curvature type. The resulting conclusions will form an evolving table updated with new propositions and invalidated propositions. This article presents the initial phase of such a table.

RESULTS AND FUTURE WORK

The results of this article will be examined in three sections. The current state of the table obtained from experiments will be interpreted first. Subsequently, the conclusions reached with each paneling approach will be presented, followed by a general assessment of the prospects for further research. When experiments were conducted on surfaces with different curvatures using various paneling attempts and evaluated against defined criteria, the following conclusions were reached (Table 2):

GEOMETRY	SPHERICAL			HYPERBOLIC			DEVELOPABLE			FREE-FORM								
CURVATURE	CONSTANT POSITIVE			CONSTANT NEGATIVE			CONSTANT ZERO			VARIABLE NEGATIVE TO ZERO			VARIABLE ZERO TO POSITIVE			VARIABLE NEGATIVE TO POSITIVE		
EXAMPLE	SPHERE			HYPERBOLOID			CONE			CONOID			ELLIPSOID			TORUS		
	A	B	C	A	B	C	A	B	C	A	B	C	A	B	C	A	B	C
CONTOURING	+	-	±	+	-	±	+	±	±	-	±	-	±	-	±	±	-	±
EUCLIDEAN SPHERES	+	±	±	+	±	±	+	+	+	-	±	±	-	±	±	-	±	±
QUAD EXTENSION										+	±	-	+	±	-	+	±	-
MARCHING CUBES	±	±	±	±	±	±	±	±	±	±	±	±	±	±	±	±	±	±
TANGENT PLANE INTERSECTION	+	-	-	+	-	-	+	-	-	+	-	-	+	-	-	+	-	-

A Non-triangular panels are planar
B Panels are equilateral
C Panels are identical

+ Hypothesis is correct if the paneling is applied properly
± Hypothesis is partially correct
- Hypothesis is incorrect

Table 2. Application of surface paneling approaches to different curvatures

- Paneling approaches generally apply to surfaces with regular zero curvature. This result is unsurprising because surfaces with regular zero curvature can be unfolded as flat and are easier to manufacture from planar materials due to their geometric properties.
- Covering surfaces with variable curvature with identical panels is often impractical. For instance, the Euclidean spheres approach can ensure that some panels are equilateral and equal due to their geometric properties, but after a certain step, it becomes impossible to adapt the paneling to variable curvatures.
- Further research is needed into the outcomes of the cube marching approach on different surfaces. This approach suggests potential by considering the entire surface as a whole and progressing by partitioning, thereby leading to more panel varieties as curvature increases. This method has its own literature and research area in computer graphics rather than architectural paneling. Architectural paneling studies should benefit more from this neighboring field.
- Tangent plane intersection and quad extension can be used to obtain planar panels when desired. Obtaining planar panels is a fundamental expectation of this research area. Both paneling approaches typically yield planar panel results due to their geometric properties, but they may not meet other criteria.
- By experimenting with several approaches concurrently on free-form (variable curvature) surfaces, new propositions, and tables can be derived. For example, using the tangent planes approach can result in planar panels while ensuring some panels are identical by partially using Euclidean spheres for selecting the points.
- The table can be expanded with new propositions and achievements. For example, evaluating how closely the obtained panels match the source surface could be considered another achievement.

The comparison presented here serves as a beginning for a study that could develop much further. Considering the diversity of architectural surfaces and the variety of tools and methods used for paneling, the resulting comparison table is a beginning that supports asking more questions. As Pottmann et al. (2007) noted, solving all issues related to the applicability of free-form surfaces is not merely about geometry knowledge. However, a good understanding of geometry should be considered an important step for such applications.

When examining the results of the comparison, it is evident that no surface paneling approach can provide the most suitable result on all types of surfaces. Each approach responds to different design criteria in different ways. This situation parallels advanced paneling algorithms reviewed in the literature beyond the selected paneling approaches. The diversity encompassed by surface paneling as a design-research field can be expanded to include other surface types, paneling approaches, and algorithms. It can be enriched with the participation of other achievements such as material and time optimization, beyond the states of being flat, equilateral, or equal. The result obtained from applying a specific paneling approach depends not only on surface curvature but also on how the surface is created and how the paneling approach is applied.

To prevent complex geometric operations from becoming pattern catalogs, their purposes, meanings, and contributions to design must be questioned. Interdisciplinary subjects like architectural paneling should be subjects that architects must be knowledgeable and opinionated about, rather than just services demanded like engineering solutions. In-depth examinations and decision-making processes, reduced to specific physical achievements, can only be made meaningful by architects. This can only be achieved with a valid and up-to-date understanding of architectural geometry. The continuity of architecture education's sensitive relationship with mathematics and geometry depends on developing contemporary architectural geometry and mathematics education in parallel with design studios. The path opened by young architects over the past decade can guide new generations by leading new syntheses.

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